

# ISO/IEC Guide 98-3 Supplement 1:2008-11 (E)

## Uncertainty of measurement - Part 3: Guide to the expression of uncertainty in measurement (GUM:1995) - Supplement 1: Propagation of distributions using a Monte Carlo method

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